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Classical functions corresponding to given quantum operators†

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Abstract. The problem of obtaining the classical function which corresponds to a given quantum operator is discussed. Its application to the phase-space distribution functions and to the ordering of an operator is briefly considered.

1. Introduction

In discussing the relation between the quantum-mechanical and the classical description one is many times required to determine the operator which corresponds to a given classical function. We shall consider the canonical coordinates and momenta as the basic dynamical quantities. In the quantum-mechanical case these satisfy the commutation relations‡

$$\left.\begin{aligned}[\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] &= 0 \\ [\hat{q}_i, \hat{p}_j] &= i\delta_{ij}\end{aligned}\right\}. \tag{1.1}$$

On account of this non-commutativity in the quantum-mechanical case, the association of operators with given classical functions is often ambiguous. Many rules are proposed which give a definite procedure for obtaining the quantum operator which corresponds to the given classical function. Among those which have been discussed at length are the following (Shewell 1959, McCoy 1932, Mehta 1964, Kubo 1964, Cohen 1965, Daughaday and Nigam 1965): (i) Weyl's rule, (ii) standard ordering, (iii) normal ordering, (iv) anti-normal ordering.

If we consider a system with only one degree of freedom, and assume that the given classical function $F(q,p)$ has a double Fourier integral representation

$$F(q,p) = \iint \gamma(\tau,\theta) \exp\{i(\tau q + \theta p)\}\, d\tau\, d\theta \tag{1.2}$$

then the operator $\hat{F}_{\mathrm{W}}$ in Weyl's rule, which corresponds to $F(q,p)$, is obtained by replacing q and p by the operators $\hat{q}$ and $\hat{p}$ respectively on the right-hand side of (1.2):

$$\hat{F}_{\mathrm{W}} = \iint \gamma(\tau,\theta) \exp\{i(\tau\hat{q} + \theta\hat{p})\}\, d\tau\, d\theta. \tag{1.3}$$

It can be shown that the operator which corresponds to $q^n p^m$ according to this formula is the coefficient of $\{(m+n)!/m!n!\}\lambda^m\mu^n$ in the expansion of $(\mu\hat{q}+\lambda\hat{p})^{m+n}$, and is thus obtained by replacing q and p by the corresponding operators in the completely symmetrized form (with respect to all permutations) of $q^n p^m$.

In standard ordering the operator $\hat{F}_{\mathrm{S}}$ which corresponds to $F(q,p)$ is given by

$$\hat{F}_{\mathrm{S}} = \iint \gamma(\tau,\theta) \exp(i\tau\hat{q}) \exp(i\tau\hat{p})\, d\tau\, d\theta \tag{1.4}$$

and is obtained by ordering q and p in $F(q,p)$ in such a way that all powers of q precede all powers of p, and then by replacing these quantities by the corresponding operators.

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‡ In this paper c-number quantities are denoted by simple letters without a circumflex (e.g. q, p, v, v^*, etc.), whereas the corresponding quantum operators are denoted by a circumflex (e.g. $\hat{q}$, $\hat{p}$, $\hat{a}$, $\hat{a}^\dagger$ etc.). We have chosen units such that $\hbar = 1$.

For discussing normal and anti-normal ordering, one introduces in place of q and p two new quantities v and v^*, defined by

$$v = \frac{q+ip}{\sqrt{2}}, \qquad v^* = \frac{q-ip}{\sqrt{2}}. \tag{1.5}$$

The operators corresponding to v and v^* will be denoted by $\hat{a}$ and $\hat{a}^\dagger$, so that they satisfy the commutation relation

$$[\hat{a}, \hat{a}^\dagger] = 1. \tag{1.6}$$

If we also write

$$\alpha = \frac{\theta - i\tau}{\sqrt{2}}, \qquad \alpha^* = \frac{\theta + i\tau}{\sqrt{2}} \tag{1.7}$$

we obtain

$$i(\tau q + \theta p) = \alpha^* v - \alpha v^* \tag{1.8a}$$

and

$$i(\tau \hat{q} + \theta \hat{p}) = \alpha^* \hat{a} - \alpha \hat{a}^\dagger. \tag{1.8b}$$

The operator which corresponds to $F(q, p)$ in the normal ordering rule is then given by

$$\hat{F}_{\mathrm{N}} = \iint \gamma(\tau, \theta) \exp(-\alpha \hat{a}^\dagger) \exp(\alpha^* \hat{a})\, d\tau\, d\theta \tag{1.9}$$

and the one in the anti-normal ordering rule is given by

$$\hat{F}_{\mathrm{A}} = \iint \gamma(\tau, \theta) \exp(\alpha^* \hat{a}) \exp(-\alpha \hat{a}^\dagger)\, d\tau\, d\theta. \tag{1.10}$$

As mentioned earlier, these rules give definite procedures for obtaining quantum operators corresponding to given classical functions. In this paper we consider the inverse problem which is also of interest, namely that of determining the classical function corresponding to a given operator. This, in particular, has applications to the phase-space distribution functions and to the ordering of an operator. For simplicity we consider systems with only one degree of freedom. The generalization to the case of systems with any finite number of degrees of freedom is more or less straightforward.

2. The general case

We are given the operator $\hat{F}$, and our problem is to derive the expression for the corresponding classical function in various rules of association. Let us assume that we can write the operator $\hat{F}$ in the form

$$\hat{F} = \iint \gamma(\tau, \theta) \exp\{i(\tau \hat{q} + \theta \hat{p})\}\, d\tau\, d\theta. \tag{2.1}$$

The Baker–Housdorff identity, which holds for any two operators $\hat{A}$ and $\hat{B}$ which commute with their commutator, reads (Messiah 1961)

$$\exp(\hat{A} + \hat{B}) = \exp(\hat{A}) \exp(\hat{B}) \exp(-\tfrac{1}{2}[\hat{A}, \hat{B}]). \tag{2.2}$$

From (2.1), (2.2) and (1.8b) we obtain

$$\hat{F} = \iint \gamma(\tau, \theta) \exp(\tfrac{1}{2} i\tau\theta) \exp(i\tau \hat{q}) \exp(i\theta \hat{p})\, d\tau\, d\theta \tag{2.3}$$

$$= \iint \gamma(\tau, \theta) \exp(-\tfrac{1}{2}|\alpha|^2) \exp(-\alpha \hat{a}^\dagger) \exp(\alpha^* \hat{a})\, d\tau\, d\theta \tag{2.4}$$

$$= \iint \gamma(\tau, \theta) \exp(\tfrac{1}{2}|\alpha|^2) \exp(\alpha^* \hat{a}) \exp(-\alpha \hat{a}^\dagger)\, d\tau\, d\theta. \tag{2.5}$$

Relations (2.1), (2.3)–(2.5) then immediately give the required classical functions $F_{\mathrm{W}}(q, p)$, $F_{\mathrm{S}}(q, p)$, $F_{\mathrm{N}}(v, v^*)$ and $F_{\mathrm{A}}(v, v^*)$ which correspond to the operator $\hat{F}$ in the Weyl's, standard, normal and anti-normal ordering† rules respectively:

$$F_{\mathrm{W}}(q,p) = \iint \gamma(\tau,\theta)\exp\{i(\tau q+\theta p)\}\,d\tau\,d\theta \tag{2.6}$$

$$F_{\mathrm{S}}(q,p) = \iint \gamma(\tau,\theta)\exp(\tfrac{1}{2}i\tau\theta)\exp\{i(\tau q+\theta p)\}\,d\tau\,d\theta \tag{2.7}$$

$$F_{\mathrm{N}}(v,v^*) = \iint \gamma(\tau,\theta)\exp\{-\tfrac{1}{4}(\tau^2+\theta^2)\}\exp\{i(\tau q+\theta p)\}\,d\tau\,d\theta \tag{2.8}$$

and

$$F_{\mathrm{A}}(v,v^*) = \iint \gamma(\tau,\theta)\exp\{\tfrac{1}{4}(\tau^2+\theta^2)\}\exp\{i(\tau q+\theta p)\}\,d\tau\,d\theta. \tag{2.9}$$

To obtain the expression for $\gamma(\tau, \theta)$, we use the relation‡ (see, for example, Kubo 1964, Imre *et al.* 1967)

$$\frac{1}{2\pi}\operatorname{Tr}\left[\exp\{i(\tau\hat{q}+\theta\hat{p})\}\right] = \delta(\tau)\delta(\theta) \tag{2.10}$$

where δ is the Dirac delta function. From (2.1), (2.2) and (2.10) we then obtain, after some simplification,

$$\frac{1}{2\pi}\operatorname{Tr}[\hat{F}\exp\{-i(\tau\hat{q}+\theta\hat{p})\}] = \gamma(\tau,\theta). \tag{2.11}$$

From equations (2.2), (2.6)–(2.9) and (2.11) we then obtain

$$\gamma_{\mathrm{W}}(\tau,\theta) \equiv \gamma(\tau,\theta) = \frac{1}{2\pi}\operatorname{Tr}[\hat{F}\exp\{-i(\tau\hat{q}+\theta\hat{p})\}] \tag{2.12}$$

$$\gamma_{\mathrm{S}}(\tau,\theta) \equiv \exp(\tfrac{1}{2}i\tau\theta)\gamma(\tau,\theta) = \frac{1}{2\pi}\operatorname{Tr}\{\hat{F}\exp(-i\theta\hat{p})\exp(-i\tau\hat{q})\} \tag{2.13}$$

$$\gamma_{\mathrm{N}}(\tau,\theta) \equiv \exp(-\tfrac{1}{2}|\alpha|^2)\gamma(\tau,\theta) = \frac{1}{2\pi}\operatorname{Tr}\{\hat{F}\exp(-\alpha^*\hat{a})\exp(\alpha\hat{a}^\dagger)\} \tag{2.14}$$

$$\gamma_{\mathrm{A}}(\tau,\theta) \equiv \exp(\tfrac{1}{2}|\alpha|^2)\gamma(\tau,\theta) = \frac{1}{2\pi}\operatorname{Tr}\{\hat{F}\exp(\alpha\hat{a}^\dagger)\exp(-\alpha^*\hat{a})\}. \tag{2.15}$$

Here α is given by (1.7). On taking the inverse Fourier transforms of (2.12)–(2.15) we obtain formally the required functions F_{W}, F_{S}, F_{N} and F_{A} which correspond to the given operator $\hat{F}$ in various rules of association.

Equations (2.12)–(2.15) give a definite way of obtaining the Fourier transforms of $F_{\mathrm{W}}(q, p)$, $F_{\mathrm{S}}(q, p)$, $F_{\mathrm{N}}(v, v^*)$ and $F_{\mathrm{A}}(v, v^*)$. However, the calculations in specific cases may become quite complicated, and sometimes one even has to consider generalized functions for this purpose. Occasionally some alternative simpler forms are available for F_{W}, F_{S}, F_{N} and F_{A}. We consider in the next section the case of normal and anti-normal ordering. Weyl's rule of association has been discussed by Kubo (1964), and the case of standard ordering is not very different. It must be emphasized that, even for a well-behaved operator $\hat{F}$, the associated classical function in general can only be interpreted in terms of generalized functions.

† For convenience we express the functions F_{N} and F_{A} as functions of v and v^* rather than of q and p.

‡ Relation (2.10) may also be derived by using coherent states (see equations (3.1) and (3.2) below). Thus from (1.8), (2.2), (3.1) and (3.2) it is seen that

$$\begin{aligned}\operatorname{Tr}[\exp\{i(\tau\hat{q}+\theta\hat{p})\}] &= \operatorname{Tr}\left\{\frac{1}{\pi}\exp(-\tfrac{1}{2}|\alpha|^2)\exp(-\alpha\hat{a}^\dagger)\,(\exp)\,\alpha^*\hat{a}\int |v\rangle\langle v|\,d^2v\right\}\\ &= \frac{1}{\pi}\exp(-\tfrac{1}{2}|\alpha|^2)\int \exp(\alpha^*v-\alpha v^*)\,d^2v\end{aligned}$$

from which (2.10) follows immediately.

3. Normal and anti-normal ordering

In this section we make use of the coherent states (Glauber 1963, Klauder 1960) to obtain $F_N(q, p)$ and $F_A(q, p)$ which correspond to the operator $\hat{F}$ in the normal and the anti-normal ordering rules respectively. The coherent states are defined as the normalized eigenstates of the operator $\hat{a}$:

$$\hat{a}|v\rangle = v|v\rangle \tag{3.1}$$

and satisfy the completeness relation

$$\frac{1}{\pi}\int |v\rangle\langle v|\, d^2v = 1 \tag{3.2}$$

where $d^2v \equiv d(\mathscr{R}v)d(\mathscr{I}v)$.

From (2.14) and (3.2) we obtain

$$\begin{aligned}\gamma_N(\tau, \theta) &= \frac{1}{2\pi^2}\mathrm{Tr}\{\hat{F}\exp(-\alpha^*\hat{a})\int |v\rangle\langle v|\, d^2v \exp(\alpha\hat{a}^\dagger)\} \\ &= \frac{1}{2\pi^2}\int \langle v|\hat{F}|v\rangle \exp(\alpha v^* - \alpha^* v)\, d^2v.\end{aligned} \tag{3.3}$$

Hence, on taking the inverse Fourier transform of (3.3), we obtain

$$F_N(v, v^*) = \langle v|\hat{F}|v\rangle \tag{3.4}$$

a relation which can also be inferred directly.

Finally, from (2.9) and (2.15) it is easy to verify that $F_A(v, v^*)$ is the diagonal coherent-state representation of the operator $\hat{F}$:

$$\hat{F} = \int F_A(v, v^*)|v\rangle\langle v|\, d^2v \tag{3.5}$$

where v is given by (1.5) and $d^2v \equiv \frac{1}{2}dq\, dp$. A precise method for obtaining the diagonal coherent-state representation has been discussed elsewhere (Mehta 1967, Klauder *et al.* 1965, Miller and Mishkin 1967) and may be used to obtain $F_A(v, v^*)$.

In the next two sections we consider some applications of this formalism.

4. Phase-space distribution functions

Let $\hat{\rho}$ be the density operator describing the statistical state of the system. The associated phase-space distribution function† $\Phi_W(q, p)$ in Weyl's rule of association satisfies the relation

$$\iint \Phi_W(q, p)\exp\{-i(\tau q + \theta p)\}\, dq\, dp = \mathrm{Tr}[\hat{\rho}\exp\{-i(\tau\hat{q} + \theta\hat{p})\}]. \tag{4.1}$$

The distribution function Φ_W has the property that, given any operator $\hat{F}$, we can write (Moyal 1949)

$$\mathrm{Tr}(\hat{\rho}\hat{F}) = \iint \Phi_W(q, p)F_W(q, p)\, dq\, dp \tag{4.2}$$

where $F_W(q, p)$ is the classical function which corresponds to the operator $\hat{F}$ in Weyl's rule of association.

If we compare (4.1) with (2.12) and use (2.6), it is readily seen that

$$\Phi_W(q, p) \equiv \frac{1}{2\pi}\rho_W(q, p). \tag{4.3}$$

† $\Phi_W(q, p)$ is essentially the Wigner distribution function (see Wigner 1932, Moyal 1949).

In a similar manner, if Φ_N and Φ_A denote the phase-space distributions in the normal ordering and the anti-normal ordering rules (Mehta 1964, Mehta and Sudarshan 1965, Kano 1965) of association, respectively, then

$$\int \Phi_N(v, v^*) \exp(\alpha v^* - \alpha^* v)\, d^2 v = \mathrm{Tr}\{\hat{\rho} \exp(\alpha \hat{a}^\dagger) \exp(-\alpha^* \hat{a})\} \tag{4.4}$$

$$\int \Phi_A(v, v^*) \exp(\alpha v^* - \alpha^* v)\, d^2 v = \mathrm{Tr}\{\hat{\rho} \exp(-\alpha^* \hat{a}) \exp(\alpha \hat{a}^\dagger)\}. \tag{4.5}$$

From equations (4.4), (4.5), (2.14), (2.15), (2.8) and (2.9) we find that†

$$\Phi_A(v, v^*) \equiv \frac{1}{\pi} \rho_N(v, v^*) \tag{4.6}$$

$$\Phi_N(v, v^*) \equiv \frac{1}{\pi} \rho_A(v, v^*). \tag{4.7}$$

We did not consider the anti-standard ordering rule of association in the above discussion. However, it may easily be verified that

$$\Phi_S(q, p) \equiv \frac{1}{2\pi} \rho_{AS}(q, p) \tag{4.8}$$

and

$$\Phi_{AS}(q, p) \equiv \frac{1}{2\pi} \rho_S(q, p) \tag{4.9}$$

where $\rho_{AS}(q, p)$ denotes the classical function which corresponds to the operator $\hat{\rho}$ in the anti-standard ordering rule of association. $\Phi_{AS}(q, p)$ is the corresponding phase-space distribution function. By anti-standard ordering, we mean here the arrangement in which all powers of $\hat{p}$ precede all powers of $\hat{q}$. Thus, for example, the operator $\hat{F}_{AS}$ corresponding to $F(q, p)$ of equation (1.2) is given by

$$\hat{F}_{AS} = \iint \gamma(\tau, \theta) \exp(i\theta\hat{p}) \exp(i\tau\hat{q})\, d\tau\, d\theta. \tag{4.10}$$

We thus find that the problem of obtaining the phase-space distribution function for certain rules of association is equivalent to obtaining the classical function corresponding to the given density operator for the 'reciprocal' rule of association.

5. Ordering of an operator

Suppose we are interested in obtaining the normally ordered form of an operator $\hat{F}$. By definition, an operator $\hat{G}(a, a^\dagger)$ is said to be in the normally ordered form if

$$\hat{G}(a, a^\dagger) = :\hat{G}(a, a^\dagger): \tag{5.1}$$

where $:\hat{G}:$ denotes the normal ordering operation on G, i.e. the operation of arranging all powers of $\hat{a}^\dagger$ to the left of all powers of $\hat{a}$ in the expansion of $\hat{G}$, without making use of the commutation relation (1.6). Thus, for example, $:a\, a^\dagger: \equiv :a^\dagger a: = a^\dagger a$. It is now obvious that if $F_N(v, v^*)$ is the classical function which corresponds to the operator $\hat{F}$ in the normal ordering rule of association, then

$$\hat{F} = :\hat{F}_N(a, a^\dagger): \tag{5.2}$$

where $\hat{F}_N(a, a^\dagger)$ is obtained by replacing v by $\hat{a}$ and v^* by $\hat{a}^\dagger$ in $F_N(v, v^*)$. Relation (5.2) can thus be used to express any operator in the normally ordered form.

In an analogous manner, if we denote by " " the anti-normal ordering operation, we obtain

$$\hat{F} = \text{“}\hat{F}_A(a, a^\dagger)\text{”} \tag{5.3}$$

† A result essentially similar to (4.6) and (4.7) has also been derived by Lax and Louisell (1967).

and this relation then gives the anti-normally ordered form of $\hat{F}$. Here $F_A(v, v^*)$ is the classical function which corresponds to the operator $\hat{F}$ in the anti-normal ordering rule.

Similarly, $F_W(q, p)$ and $F_S(q, p)$ may be used to obtain the completely symmetrized form of $\hat{F}$ and the standard ordered form of $\hat{F}$, respectively.

As an example, we consider the normally and anti-normally ordered forms of the operator

$$\hat{F} = \exp(-\lambda \hat{a}^\dagger \hat{a}). \tag{5.4}$$

From (3.4) we find that

$$F_N(v, v^*) = \langle v | \exp(-\lambda \hat{a}^\dagger \hat{a}) | v \rangle. \tag{5.5}$$

If we insert the identity operator

$$\sum_n |n\rangle \langle n| = 1 \tag{5.6}$$

in (5.5) and also use the relation

$$\langle n|v\rangle = \exp(-\tfrac{1}{2}|v|^2)\frac{v^n}{(n!)^{1/2}} \tag{5.7}$$

where $|n\rangle$ is the number state (the eigenstate of $\hat{a}^\dagger\hat{a}$ with eigenvalue n), we obtain

$$\begin{aligned} F_N(v, v^*) &= \sum_n \langle v| \exp(-\lambda \hat{a}^\dagger \hat{a})|n\rangle \langle n|v\rangle \\ &= \exp\{-(1-\mathrm{e}^{-\lambda})v^* v\}. \end{aligned} \tag{5.8}$$

From (5.2) and (5.8) we thus obtain (see also Louisell 1965, Mandel 1966)

$$\exp(-\lambda \hat{a}^\dagger \hat{a}) = :\exp\{-(1-\mathrm{e}^{-\lambda})\hat{a}^\dagger \hat{a}\}:. \tag{5.9}$$

It is worth mentioning that the relation (5.9) may also be used to obtain the anti-normally ordered form of $\exp(-\lambda \hat{a}^\dagger \hat{a})$. From (5.9) it follows that for any two operators $\hat{d}$ and $\hat{c}$ which satisfy the commutation relation

$$[\hat{d}, \hat{c}] = 1 \tag{5.10}$$

the relation

$$\exp(\lambda \hat{c}\hat{d}) = \sum_{n=0}^{\infty} \frac{1}{n!}\{-(1-\mathrm{e}^{\lambda})\}^n \hat{c}^n \hat{d}^n \tag{5.11}$$

must hold. We now substitute

$$\hat{d} = \hat{a}^\dagger, \qquad \hat{c} = -\hat{a} \tag{5.12}$$

in (5.11) and obtain

$$\begin{aligned} \exp(-\lambda \hat{a}\hat{a}^\dagger) &= \sum_{n=0}^{\infty} \frac{1}{n!}(1-\mathrm{e}^{\lambda})^n \hat{a}^n \hat{a}^{\dagger n} \\ &\equiv \text{“}\exp\{(1-\mathrm{e}^{\lambda})\hat{a}^\dagger \hat{a}\}\text{”}. \end{aligned} \tag{5.13}$$

Relation (5.13) may be rewritten in the form

$$\exp(-\lambda \hat{a}^\dagger \hat{a}) = \mathrm{e}^{\lambda}\, \text{“}\exp\{-(\mathrm{e}^{\lambda}-1)\hat{a}^\dagger \hat{a}\}\text{”}. \tag{5.14}$$

Throughout the above discussion we have considered systems with only one degree of freedom. As an illustration of applications to systems with several degrees of freedom, we derive below the normally and anti-normally ordered forms of the operator

$$\hat{F} = \exp\left\{-\sum_{i=1}^{N} \sum_{j=1}^{N} (\hat{a}_i^\dagger + \alpha_i^*)\lambda_{ij}(\hat{a}_j + \beta_j)\right\}. \tag{5.15}$$

We consider the case when the matrix $\{\lambda_{ij}\}$ can be diagonalized by a similarity transformation

$$S\lambda S^{-1} = \Lambda \tag{5.16}$$

where Λ is diagonal. The operators $\hat{a}_i$ and $\hat{a}_i^\dagger$ satisfy the commutation relations

$$[a_i, a_j] = [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0 \tag{5.17a}$$

$$[a_i, \hat{a}_j^\dagger] = \delta_{ij}. \tag{5.17b}$$

Let us introduce two sets of operators $\hat{d}_1, \hat{d}_2, ..., \hat{d}_n$ and $\hat{c}_1, \hat{c}_2, ..., \hat{c}_n$, defined by

$$\hat{d}_i = \sum_{j=1}^{N} S_{ij}(\hat{a}_j + \beta_j) \tag{5.18a}$$

$$\hat{c}_i = \sum_{j=1}^{N} S_{ji}^{-1}(\hat{a}_j^\dagger + \alpha_j^*) \tag{5.18b}$$

where S is the matrix which diagonalizes λ.

From (5.17) and (5.18) we find that

$$[\hat{c}_i, \hat{c}_j] = [\hat{d}_i, \hat{d}_j] = 0 \tag{5.19a}$$

$$[\hat{d}_i, \hat{c}_j] = \delta_{ij} \tag{5.19b}$$

and from (5.11) we then obtain

$$\exp\left(-\sum_{i=1}^{N} \Lambda_i \hat{c}_i \hat{d}_i\right) = \prod_{i=1}^{N} \exp(-\Lambda_i \hat{c}_i \hat{d}_i)$$

$$= \prod_{i=1}^{N} \left[\sum_{n=0}^{\infty} \frac{1}{n!} \{-(1-\mathrm{e}^{-\Lambda_i})\}^n \hat{c}_i^n \hat{d}_i^n\right]. \tag{5.20}$$

Now, it follows from (5.16) and (5.18) that

$$\sum_{i=1}^{N} \sum_{j=1}^{N} (\hat{a}_i^\dagger + \alpha_i^*)\lambda_{ij}(\hat{a}_j + \beta_j) = \sum_{i=1}^{N} \Lambda_i \hat{c}_i \hat{d}_i. \tag{5.21}$$

Finally, from (5.18), (5.20) and (5.21) we obtain the required result

$$\exp\left\{-\sum_{i=1}^{N} \sum_{j=1}^{N} (\hat{a}_i^\dagger + \alpha_i^*)\lambda_{ij}(\hat{a}_j + \beta_j)\right\}$$

$$= :\exp\left\{-\sum_{i=1}^{N} \sum_{j=1}^{N} (\hat{a}_i^\dagger + \alpha_i^*)(I - \mathrm{e}^{-\lambda})_{ij}(\hat{a}_j + \beta_j)\right\}: \tag{5.22}$$

where I is the identity matrix.

In a similar manner one can show that†

$$\exp\left\{-\sum_{i=1}^{N} \sum_{j=1}^{N} (\hat{a}_i + \alpha_i^*)\lambda_{ij}(\hat{a}_j + \beta_j)\right\}$$

$$= \text{“}\exp\left\{\mathrm{Tr}\,\lambda - \sum_{i=1}^{N} \sum_{j=1}^{N} (a_i^\dagger + \alpha_i^*)(\mathrm{e}^{\lambda} - I)_{ij}(\hat{a}_j + \beta_j)\right\}\text{”}. \tag{5.23}$$

Relations (5.22) and (5.23) may be used to obtain the classical functions corresponding to the operator (5.15) in the normal and the anti-normal ordering rules. They may also be used to obtain the associated phase-space distribution functions in the case when (5.15), with proper normalization, is the density operator. In particular, equation (5.23) gives immediately the diagonal coherent-state representation of (5.15).

† In the case of systems with one degree of freedom ($N = 1$) the relations (5.22) and (5.23) have been derived earlier (see Mehta 1967, Wilcox 1967).

Note added in proof. Recently Agarwal and Wolf (1968) have considered some related problems concerning a unified treatment of various rules of associations (see also Cohen 1965).

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